From symmetry to geometry

Meetkunde speelt al sinds mensenheugenis een belangrijke rol in de natuurkunde. In het eerste deel van deze Engelstalige serie artikelen over meetkunde in de natuurkunde legt Efe Utku uit wat meetkunde precies is, en hoe het begrip “symmetrie” een centrale rol daarin speelt.

Figure 1. Geometria (1565). An allegorical depiction of geometry engraved on paper by Cornelis Cort. Currently stored in the print room of Rijksmuseum in Amsterdam. Retrieved under CC0 1.0 via [Look and Learn].

Before geometry and physics ever existed as an intellectual endeavour, there was pattern-seeking. This activity was not something performed for the sake of it, nor was it performed by members of a distinguished human society at a distinguished time period. Members of different societies who lived at different times, solely due to sharing a common evolutionary inheritance, just saw patterns in nature and communicated their observations. Over time, this activity stayed intact but the vocabulary used to describe the patterns gradually evolved and ways of describing patterns changed. What did not change was that the patterns were intuitively thought of as residing in a space, such that a pattern could be transported from one place to another to compare how or when an observed regularity remains present. Thus, an understanding of "sameness" always came alongside a space. This theme, which I shall call "thinking-in-space", today sits at the centre of how physics is practiced – and its vocabulary is that of geometry.

However, since the vocabulary is ever evolving, to non-practioners of physics, its language often does not ring a bell at first and can act as a barrier that arises between the observations of scientists and the patterns other humans see in nature. This creates a distance between how scientists talk about the world and how everyone else experiences it. Do not worry if the following sentence does not make sense at the moment, but to give an example of this geometric vocabulary of physics, today elementary particles are thought of as "irreducible unitary representations of the Poincaré group". In a series of articles, we will see why geometric reasoning underpins modern physics, how this language barrier arose despite the fact that geometry literally means "earth-measurement", and try to gradually build an understanding of the technical sentence I just mentioned – and many similar ones. With these discussions I will invite you to appreciate the power of thinking-in-space.

Space and symmetry as organizational tools

Before we start with developing the concept of geometry, let us first consider what "thinking-in-space" might even mean. Surely, we are talking about some form of "spatial thinking" activity, and in fact this is something we all do, every day. Just consider some of the words I used in the last paragraph: "barriers", I invoked a sense of "height"; "distance", that is a notion of "extension" between 2 "places"; "today", a notion of place within temporal extension. All readers of English can understand what I meant with those words, without paying any extra attention to the fact that they were thinking-in-a-space. In other words: as humans we can effortlessly think-in-spaces, and, generally speaking, have an intuitive conception of "space". The reason we think-in-space is that we usually want to put things in order. That way we can almost reflexively assign places to things, be it spatial or temporal. Right, left, up, down, before, after; these are all words we use to find our way in a space, or simply to "locate" where we are. Location is the key word. We would like to distinguish one location or event from another, spatially or temporally, so that we know where we are. This knowledge of location lets us carry out other activities in a more orderly way. With this in mind, let us give our first working definition of space: space is a tool for organizing things. Since it is easier to navigate through organized things and humans love to make things easier, humans often find themselves thinking-in-space.

How does this connect to physicists and the enterprise of physics? Let us start with a seemingly circular definition. Physics is what is practiced by physicists. You may ask, what, then, do physicists practice? Physicists, just like everyone reading this text, are humans; so their task involves an organization of some sort. What do they organize? They organize their observations of natural phenomena. That may include literally anything that happens in nature. Since obviously a lot of things happen in nature, physicists organize what they observe in nature. In that way it is easier for them to navigate through what they learn about natural phenomena or to talk about them afterwards. Stated differently: in physics practice, physicists aim to distinguish one observation about nature from another, so that they know what they are talking about. This is analogous to our previous example of “knowing where you are”, but now applied to the space that is used to organize observations of natural phenomena.

At first glance, this may seem like a rather trivial task and you may rhetorically ask: “how do physicists not know when one observation is different from another?” As it turns out, there are many observations one could perform of nature which, despite looking deceptively different, are in fact manifestations of the same phenomenon. So, one task for a physicist is to identify those differently looking observations as the same thing and get to its essence. Identification is also a key word. Humans would like to identify things, so they know whether they are the same or different. A natural question is then: “how can we identify a thing?” The short answer is: we can look at the ways a thing retains its identity. If upon being acted on, one thing “looks” the same and another doesn’t, we can use a shortcut to say that action is the identity of the thing. This turns out to be a very operational and intuitive form of identification. It is intuitive because what we are doing is essentially asserting that “symmetry” of the thing is its identity. Symmetries decide whether a thing looks the same after an action or not. If there is no action that can distinguish a thing from another, they are indeed the same. In light of this we can try to re-express what physicists do: physicists study symmetries of nature so that they can identify natural phenomena.

From practice to theory

How does geometry fit in this picture? Before talking about physics and symmetry’s relation to geometry, let us start with what we understand from the concept of geometry itself. Depending on one’s interest and prior experience with the subject, if I were to ask someone what geometry is, I would typically get answers like: (i) geometry is a branch of mathematics, (ii) geometry studies properties of shapes, or (iii) geometry makes statements about “geometric magnitudes” such as lengths and angles. All of these answers capture some aspects of geometry, and hence are partially true. However, these answers do not easily allow us to define what geometry *is*.

To get a better understanding, first let us consider the meaning of the word geometry: it is Greek for “earth-measurement” (geo-metron). It is a word which started to be used by ancient Greeks who invented the mathematical practice of geometry. However, the Greeks did not create the concept out of thin air; what they did was collect and formalize a collection of literal earth-measurement activities. That is the root of the term. So, long before the word geometry existed, ancient societies practiced geometry, but in the form of labour-based activities which solved their daily practical problems. In other words: geometry started out as a collection of technologies involving reliable measurement methods.

As examples, people used ropes to fix “straight lines” required for constructing built structures; they used knotted ropes to establish “right angles” required for marking agricultural field boundaries; or they listed tables to measure areas of land required for taxation. All of these were “geometrical” in the sense that they were used to physically measure out spatial quantities; and “technological” as the methods always worked, regardless of the place or day. Later on, ancient Greeks formalized the methods with “geometric diagrams” to show none of those practices were in fact separate, but instead they all were parts of a whole. The diagrams were very practical because they allowed them to talk about geometry. Beforehand, the tool was used and that was it. No one could argue about what the method was doing, as it just worked. Greeks, after inventing the diagrammatic practice of geometry, turned it into an intellectual activity where one could make claims about earth-measurement. So, without physically doing the labour, one could talk about its consequences regarding notions like length, area, volume, angle etc.

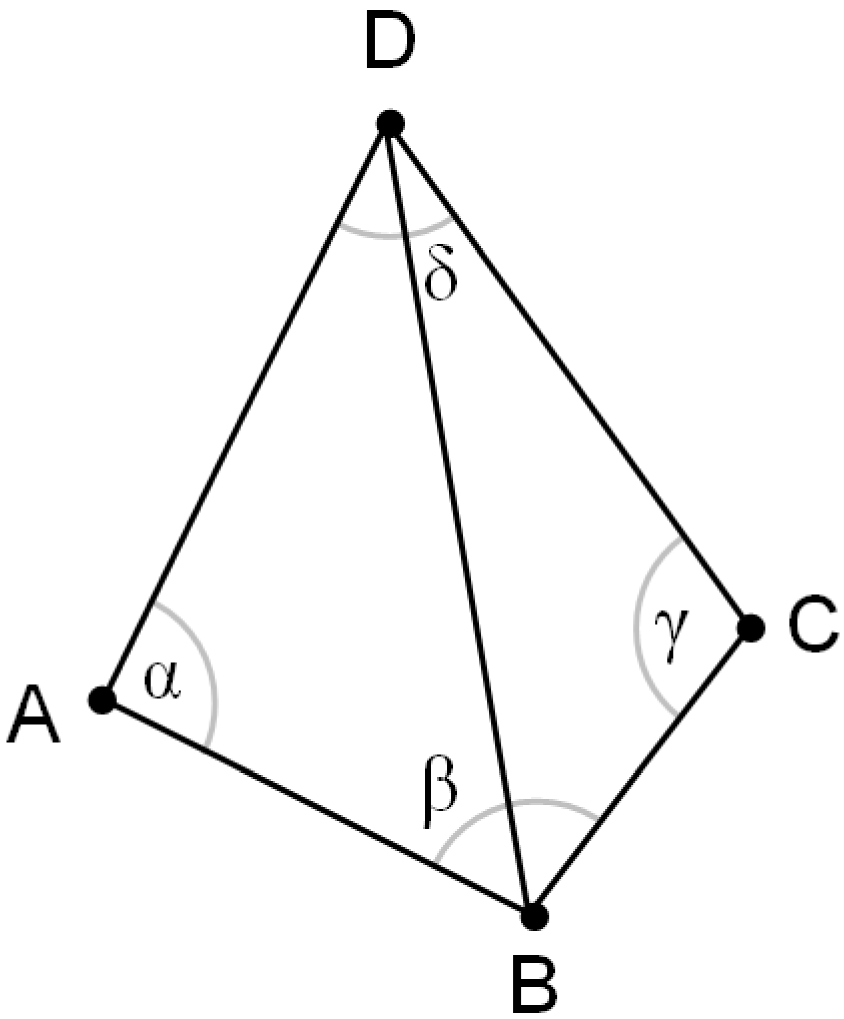

Figure 2. A geometric diagram. “Delaunay geometry” by Nü es, CC BY-SA 3.0, via Wikimedia Commons.

What is geometry?

When one is asked about “what geometry is”, this kind of diagram is typically one of the first things that comes to mind, perhaps as a consequence of early prior exposure. The answers to the question “what is geometry?” then generally revolve around concepts reminiscent of the diagrams, like shapes (triangle, circle, square) or associated magnitudes (length, area, angle). Of course, all of these are related to geometry, but the concept of geometry has much more to offer. Now that we know why these spatial concepts historically arose within geometry, let us look at why these concepts can be considered “geometrical” at all. This inquiry will lead us to a more proper definition of geometry and an understanding of why something is geometrical.

First of all, we can see that geometry has an association with space. This should be somewhat intuitive: all shapes are always thought of as residing within a space, be it a plane (a 2-dimensional space) or the ordinary 3-dimensional space. Furthermore, shapes have some defining recognizable spatial characteristics: a circle has all its points at a fixed distance from a centre point; a square has four equal sides meeting at four right angles. These characteristics derive from a sense of spatial regularity. In fact, the shapes manifest the regularities of their space. This is also why thinking-in-space is intuitive: our experience with ordinary 3-dimensional space feels orderly and intuitive. Claims about regularities of shapes (geometrical claims), like those relating to their lengths and angles, are claims about regularities of the space they reside in.

But what happens if we decide to place a familiar shape within a less intuitive space, like a sphere, the 2-dimensional surface of a ball? Can shapes retain their familiar identities? The answer is no. For example, if you place a square on a sphere, you will see that each angle of the "square" is now greater than a right angle and the interior angles will sum up to more than four right angles. You may question this claim, since the surface of the Earth is also spherical and squares on the ground appear to be ordinary "squares". This is true, but only because the square you are considering is much smaller than the Earth, so you cannot tell the difference with a square drawn on a truly flat surface. If you consider a square whose corners are located at different continents, the effect will be much more nuanced. Similarly, if a circle is placed on a sphere, its circumference will be smaller than the regular 2pr expression with the difference growing as the circle's radius increases relative to the size of the sphere.

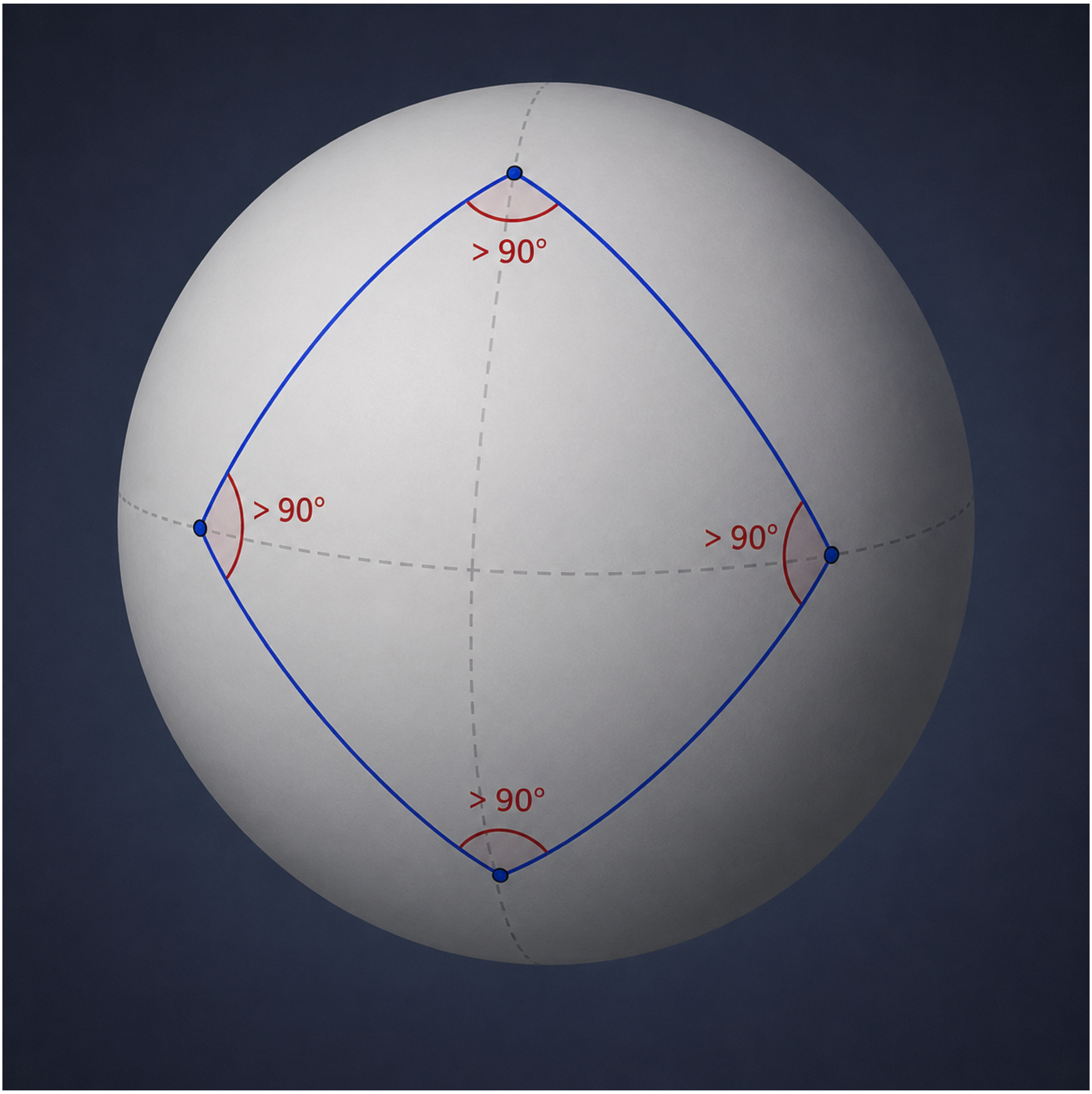

Figure 3. A square placed on a sphere. Generated with ChatGPT.

The role of symmetry

These examples show that geometric properties of shapes (or geometric objects) depend entirely on the space in which they are considered. A question arises: If properties of shapes may disappear or change, how can we even define any geometric object? This question takes us back to the concept of symmetry; we want to talk about how or when we can identify things as “the same”. Let us start with the intuitive playground and again consider a circle and square placed on a plane. Now what does it mean to be a “circle”? We can distinguish two cases: either we identify a circle as the same when after a certain set of operations we get the exact same circle, or we are interested in a “rule” that governs being a circle (“circleness”).

In the first case, if we want to have the exact same circle, what we should do is first outline the circle and check after which “transformations” we can get back the same circle. These transformations are reflections and rotations. After flipping the circle or rotating the circle around its centre, we get exactly the same circle back. We know that because the transformed circle matches the previous one, and a notion of “matching” is provided by distances measurable on the plane (see Figure 4). In the case of square, it is similar: some reflections still retain the same square, as do some rotations, but rotating the square arbitrarily or reflecting it in an arbitrary line, we cannot get the same square back. The same outlining shape can only be obtained if we rotate the square by an integer multiple of 90 degrees or if we reflect it in a horizontal, vertical or diagonal line.

These are the symmetries of “a circle” and “a square”. If we now consider the second notion of sameness, circleness and squareness, we already know that a type of reflection and a type of rotation should be symmetries of “the circle” and “the square”. But there is more: We can also freely move the shapes within the plane to get a circle and square (see Figure 5), an object that still fits the category of circleness and squareness respectively.

Mathematically, we call these transformations “rigid transformations” and they retain the lengths and angles from which we identify the objects as the same. There is also another type of transformation we can do: we can allow “scaling”. Scaling still preserves circleness and squareness but it has a cost: now absolute lengths are meaningless and only “ratios of lengths” and angles survive. Thus, if these transformations are allowed, two circles with different radii count as the same.

We can go on like this. If we add shearing and stretching, angles also do not survive. Now a circle becomes an ellipse and a square becomes a parallelogram (see Figure 6). What survives are the notions of straightness and parallelism. In this case, for example, any “four-sided figure with parallel opposite edges” counts as the same. Lastly, we can allow bending without tearing. This time straightness does not survive, and circles and squares count as the same: just closed loops. Only one notion survives and that is connectivity of their constituent points. Mathematically speaking, now we have left geometry and are in the domain of "topology".

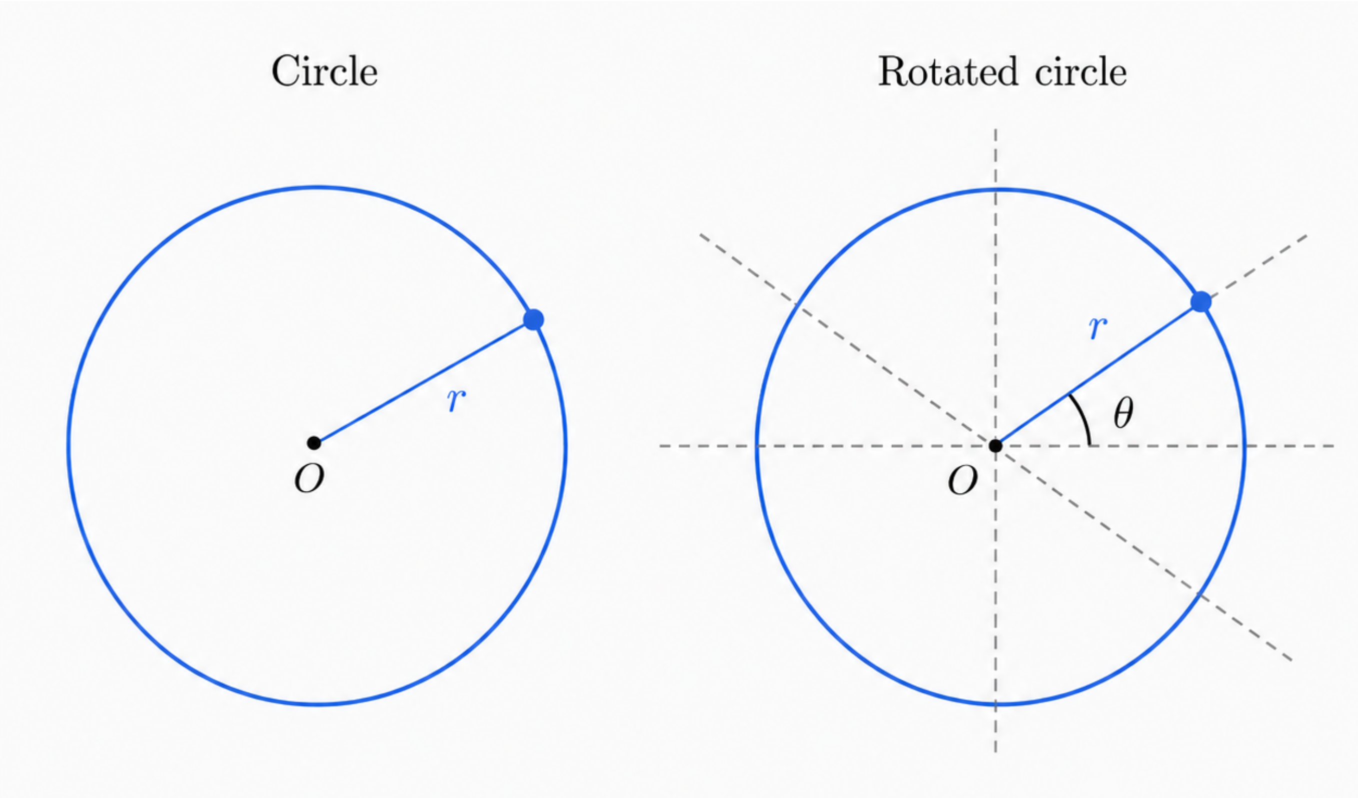

Figure 4. Rotations. A circle rotated by an angle theta around its centre is still a circle. Generated with ChatGPT.

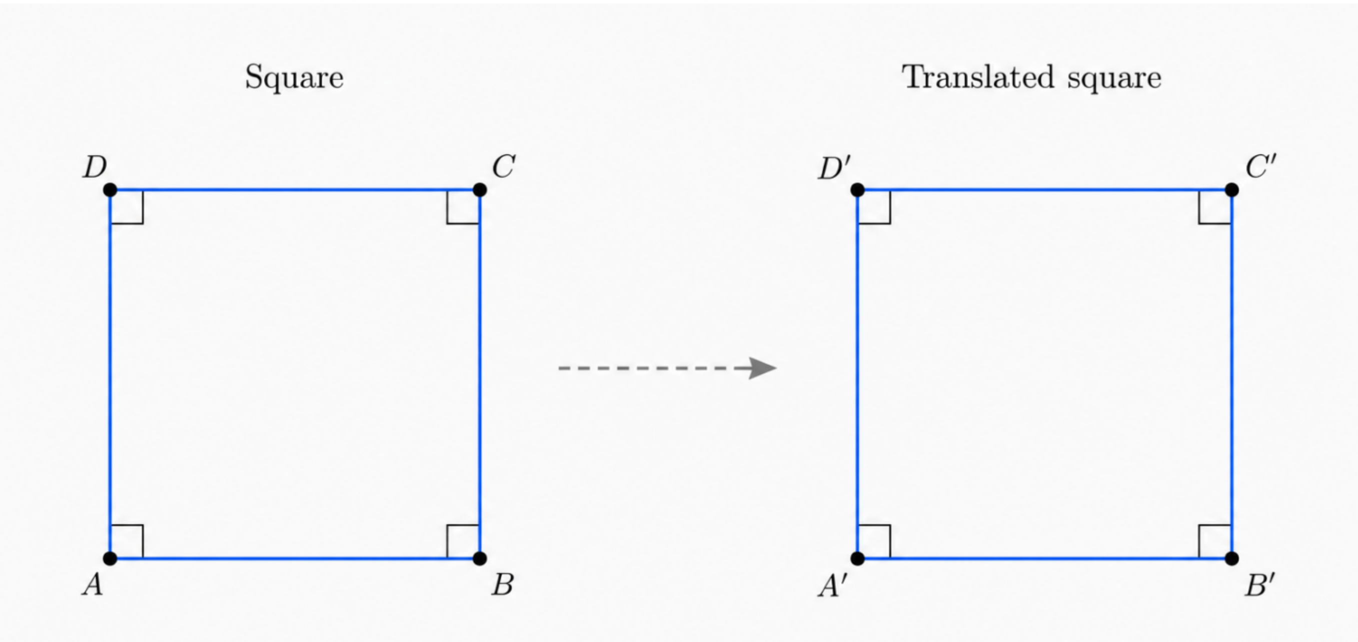

Figure 5. Translation. A square translated on the plane is still a square. Generated with ChatGPT.

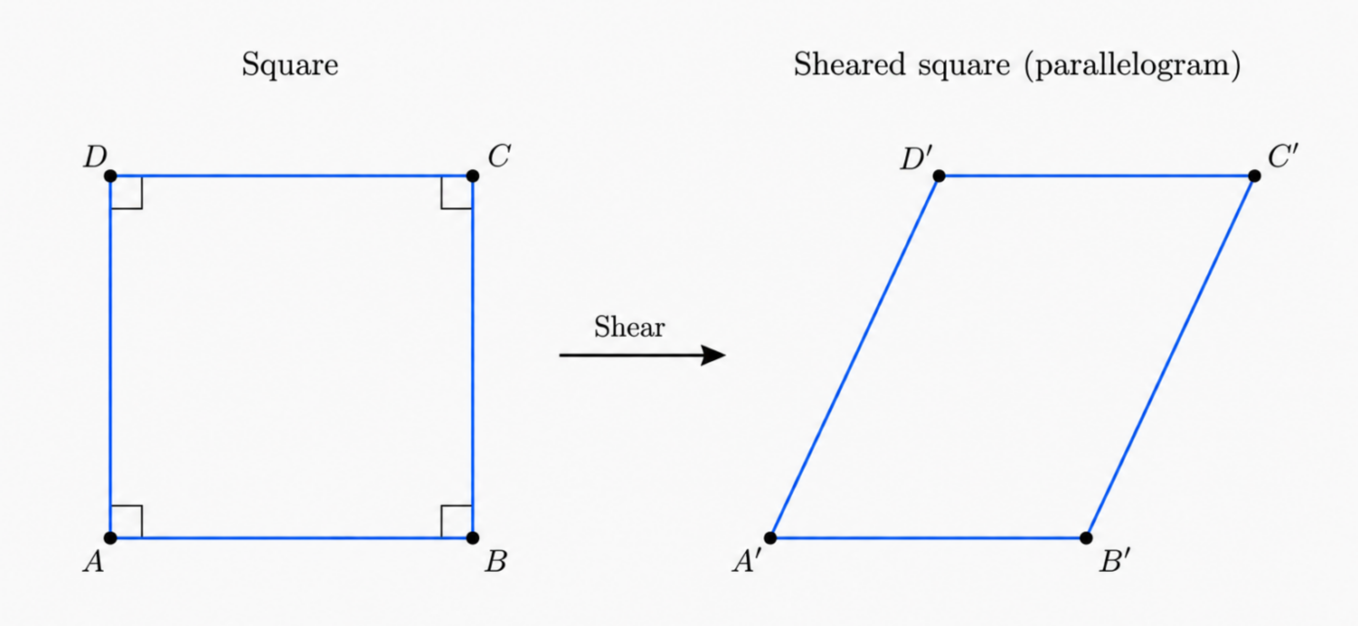

Figure 6. Shearing. A square becomes a parallelogram after a shear transform. Generated with ChatGPT.

Geometry and nature

The lesson we get out of this exercise is that sameness, and notions we use to define sameness, ultimately depend on the transformations we allow. By choosing a set of transformations, we are essentially deciding on which notions (lengths, angles, straightness, connectivity) are meaningful and notions we allow ourselves to talk about. In the context of geometry, that is, leaving aside the last example of bending, the claims one can make about geometry and its notions are determined by the set of transformations one allows themselves. Talking about ordinary lengths and angles is only meaningful if you only allow rigid transformations; talking about angles and shapes is meaningful if you only allow rigid transformations+ scaling; talking about parallelism and straightness is meaningful if you only allow rigid transformations+ scaling + shearing/ stretching. All of these domains belong to the world of geometry, but they capture different geometries since their notions differ. In the mentioned order, the respective domains are *Euclidean geometry*, *similarity geometry*, and *affine geometry*. For example, we saw by placing a “Euclidean square” on a sphere that the notion of angle does not directly translate. That is because in *spherical geometry* the Euclidean notion of straightness is lost.

What is the one common thing that permits us to call these domains “a geometry”? They study a collection of notions which remain meaningful under a set of transformations. The notions determine the geometry we talk about and notions are determined by transformations. Hence, ultimately, “transformations characterize a geometry” and to give an answer to our previous question: “geometry is the study of properties that remain invariant (unchanged) under a set of transformations, but not under bending[1].” Thought in this direction, transformations precede geometry, or stated differently: a geometry is constructed by a choice of transformations.

Having obtained this understanding of geometry, let us now close the circle and connect this back to thinking-in-space and physics. I previously stated that we think-in-space as space feels orderly, that we derive a notion of order from our experience with physical world, and that the order of space is captured by geometry. An implication of our definition of geometry is actually that order we attribute to space is a byproduct of the transformations we assume possible. In other words, it is not a property of space on its own but rather a consequence of having fixed motions that preserve identities, hence symmetries. For example, we intuitively think that no point in space is “special”, so we assume translations are possible; we assume no direction is special, so rotations are possible. For the Universe, apparently, we do not “choose” which transformations are possible for physical space and as a corollary we do not choose its geometry. That, we call nature. From observations we know that carrying a pencil between two points in space (translation) does not change the behaviour of the pencil and nor does rotating or reflecting it. However, we can measure absolute distances (not just ratios) so we know scaling is typically not a possible transformation[2]. That means that natural phenomena, or laws of physics, also possess some symmetries that we can investigate. These symmetries, the geometry of nature, will be the subject of the discussion in the next instalment in this series.

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[1] “Global” properties that remain invariant under continuous deformations (bending, stretching, twisting), like how an object’s “local” parts stay connected, are studied under the name “topology”, not geometry.

[2] Some physical systems display scale invariance, but fundamental laws of physics in general do not.